GENERALIZATIONS OF STRASSEN'S EQUATIONS FOR SECANT VARIETIES OF SEGRE VARIETIES

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Abstract. We define many new examples of modules of equations for secant varieties of Segre varieties that generalize Strassen's commutation equations [7]. Our modules of equations are obtained by constructing subspaces of matrices from tensors that satisfy various commutation properties.

1. Introduction

Let $V, A_1, ..., A_n$ be vector spaces over an algebraically closed field K of characteristic zero, and let

$$Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n) \subset \mathbb{P}(A_1 \otimes \cdots \otimes A_n)$$

denote the *Segre variety* of decomposable tensors inside $\mathbb{P}(A_1 \otimes \cdots \otimes A_n)$.

Let $X \subset \mathbb{P}V$ be a projective variety. Define $\sigma_r = \sigma_r(X)$, the *variety of secant* $\mathbb{P}^{r-1}$*'s to* X by

$$\sigma_r(X) = \overline{\cup_{x_1,...,x_r \in X} \mathbb{P}_{x_1,...,x_r}}$$

where $\mathbb{P}_{x_1,...,x_r} \subset \mathbb{P}V$ denotes the linear space spanned by $x_1, ..., x_r$ (usually a $\mathbb{P}^{r-1}$).

For applications to computational complexity, algebraic statistics and other areas, one would like to have defining equations for secant varieties of triple Segre products, in particular because the border rank r of a bilinear map $T : A^* \times B^* \to C$ is the smallest r such that $[T] \in \sigma_r(Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$. Here, and throughout this paper, "defining equations" refers to set theoretic defining equations.

Defining equations are known only for the following cases: all secant varieties of the two factor Segre (classical: these are just the $(r+1) \times (r+1)$ minors of the space of $a \times b$ matrices), the n-factor Segre itself $\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n \subset \mathbb{P}(A_1 \otimes \cdots \otimes A_n)$ (classical), its first secant variety $\sigma_2(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n)$ [3], $\sigma_3(\mathbb{P}^{a-1} \times \mathbb{P}^{b-1} \times \mathbb{P}^{c-1})$ [4], $\sigma_4(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2)$ [7], $\sigma_r(\mathbb{P}^1 \times \mathbb{P}^{b-1} \times \mathbb{P}^{c-1})$ [4] and several cases of the last nontrivial secant variety of $\mathbb{P}^2 \times \mathbb{P}^{b-1} \times \mathbb{P}^{b-1}$ when the last nontrivial secant variety is a hypersurface [7].

Segre products and their secant varieties are invariant under the action of the group $G = GL(A_1) \times \cdots \times GL(A_n)$ and thus their defining equations are best described as G-modules. In [3] we explained how one can systematically find G-modules in the ideal of the secant varieties using representation theory. We also observed that the expressions even for highest weight vectors in the modules become too complicated to write down explicitly very quickly, so there are severe limits to the systematic approach.

The equations for $\sigma_2(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n)$ may be thought of as those coming from the two factor case, that is, as minors of ordinary matrices by considering, e.g., $A \otimes B \otimes C$ as $A \otimes (B \otimes C)$ and taking the minors of the resulting $a \times bc$ matrix and permutations of such.

Strassen defined equations for $\sigma_3(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$ when $b = c$ and $a = 3$ by choosing a basis of A^* and contracting tensors to obtain subspaces of $B \otimes C$, and finding closed conditions on such

Landsberg supported by NSF grant DMS-0305829.

subspaces coming from tensors in $\sigma_3(\mathbb{P}A\times\mathbb{P}B\times\mathbb{P}C)$. From this perspective, one could look for other closed conditions on such subspaces, which is one way to view our generalizations.

Another perspective on the equations for secant varieties is that in general, if $X\subset Y$, then $\sigma_r(X)\subseteq\sigma_r(Y)$ and the equations for $\sigma_2(\mathbb{P}A\times\mathbb{P}B\times\mathbb{P}C)$ comes from the observation that $Seg(\mathbb{P}A\times\mathbb{P}B\times\mathbb{P}C)\subset Seg(\mathbb{P}A\times\mathbb{P}(B\otimes C))$.

More generally, one should look for natural varieties, whose defining equations are easily described, that contain $\sigma_r(\mathbb{P}A\times\mathbb{P}B\times\mathbb{P}C)$. From this perspective, our new equations are induced by equations of various types of varieties of subspaces of matrices that satisfy certain commutation properties.

For a partition π of d, we let $S_\pi A$ denote the corresponding irreducible $GL(A)$ module and $\Lambda_\pi A=S_{\pi'}A$ where π' is the conjugate partition to π. Our main result, theorem 4.2, may be phrased as follows:

For each r, and s sufficiently small ($s\le r/2$ if r is even, $s\le r/3$ if r is odd), we describe an explicit realization of the module

$$S_{r-s,s,s}A\otimes\Lambda_{r,s}B\otimes\Lambda_{r,s}C\subset S^{r+s}(A\otimes B\otimes C)$$

as a module of equations of $\sigma_r(\mathbb{P}A^*\otimes\mathbb{P}B^*\otimes\mathbb{P}C^*)$, and each of these modules is independent in the ideal of $\sigma_r(\mathbb{P}A^*\otimes\mathbb{P}B^*\otimes\mathbb{P}C^*)$.

(We often reverse the roles of vector spaces with their duals to eliminate $*$-s from the modules defining equations.)

The determination of the generators of the ideal of $\sigma_3(\mathbb{P}^{a-1}\times\mathbb{P}^{b-1}\times\mathbb{P}^{c-1})$ in [4] relies on a computer calculation to prove the $\sigma_3(\mathbb{P}^2\times\mathbb{P}^2\times\mathbb{P}^2)$ case is generated by Strassen's equations, and this computer calculation was originally announced in [1]. In §5 we give a computer free proof that the modules inherited from Strassen's equations give set-theoretic defining equations for $\sigma_3(\mathbb{P}^{a-1}\times\mathbb{P}^{b-1}\times\mathbb{P}^{c-1})$. A key point in our proof is the irreducibility of the variety of pairs of commuting matrices. This irreducibility fails for triples of commuting matrices. The following natural question appears to be closely related to our problem: *Find equations that characterize the irreducible component of the variety of triples of commuting matrices containing triples of regular semisimple matrices as an open subset.*

One can put our investigation in the broader context of the study of the geometry of orbit closures: let G be a complex semi-simple group, let $V=V_l$ be an irreducible G module of highest weight l. Then Kostant showed that the ideal of the closed orbit $G.[v_l]=G/P\subset\mathbb{P}V$ is generated in degree two by ${V_{2l}}^\perp\subset S^2V^*$. If we consider other G-varieties in $\mathbb{P}V$, what can we say about their defining equations?

1.1. **Overview.** In §2 we review inheritance and remark that using *subspace varieties* (defined in the section) the problem of determining defining equations of secant varieties of Segre varieties is reduced to the case of $\sigma_r(\mathbb{P}^{r-1}\times\cdots\times\mathbb{P}^{r-1})$. In §3 we review Strassen's equations for $\sigma_r(\mathbb{P}^2\times\mathbb{P}^{b-1}\times\mathbb{P}^{b-1})$, reformulate them more invariantly, and describe the modules of equations associated to his conditions. In §4 we generalize Strassen's equations and state our main result, theorem 4.2. In §5 we show that our generalizations significantly reduce the problem of determining defining equations in some cases, in particular solving it when $r=3$. In §6 we generalize our approach further and put it in a larger context, that of a class of contractions we call *coercive*. Finally in §7 we finish the proof of theorem 4.2, showing that many of the new modules of equations we defined are indeed nontrivial.

2. Inheritance and subspace varieties

2.1. **Inheritance.** We review some facts from [3]. The varieties $\sigma_r(\mathbb{P}A_1^*\times\cdots\times\mathbb{P}A_n^*)$, are invariant under the action of the group $G=GL(A_1)\times\cdots\times GL(A_n)$. Thus their ideals are given

by direct sums of irreducible submodules $S_{\pi_1}A_1 \otimes \cdots \otimes S_{\pi_n}A_n \subset S^d(A_1 \otimes \cdots \otimes A_n)$, where each π_j is a partition of d. If $\dim A_j = a_j$ then π_j can have at most a_j parts. We let $l(\pi)$ denote the number of parts of the partition π. For a variety $X \subset \mathbb{P}V$, we let $I_d(X) \subset S^dV^*$ denote the component of the ideal of X in degree d.

Proposition 2.1. [3] *If an irreducible module $S_{\mu_1}A_1 \otimes \cdots \otimes S_{\mu_n}A_n \subset I_d(\sigma_r(\mathbb{P}A_1^* \times \cdots \times \mathbb{P}A_n^*))$, then for all vector spaces $A'_j \supseteq A_j^*$, we have $(S_{\mu_1}A'_1 \otimes \cdots \otimes S_{\mu_n}A'_n)^* \subset I_d(\sigma_r(\mathbb{P}A'_1 \times \cdots \times \mathbb{P}A'_n))$.*

Moreover, a module $(S_{\mu_1}A'_1 \otimes \cdots \otimes S_{\mu_n}A'_n)^*$ where the length of each μ_j is at most a_j is in $I_d(\sigma_r(\mathbb{P}A'_1 \times \cdots \times \mathbb{P}A'_n))$ iff the corresponding module is in $I_d(\sigma_r(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n))$.

Thus a copy of a module $S_{\mu_1}A_1 \otimes \cdots \otimes S_{\mu_n}A_n$ will be in $I(\sigma_r(\mathbb{P}^{r-1} \times \cdots \times \mathbb{P}^{r-1}))$ iff the corresponding copy of the module $S_{\mu_1}\mathbb{C}^{l(\mu_1)} \otimes \cdots \otimes S_{\mu_n}\mathbb{C}^{l(\mu_n)}$ is in the ideal of $\sigma_r(\mathbb{P}^{l(\mu_1)-1} \times \cdots \times \mathbb{P}^{l(\mu_n)-1})$.

2.2. **Subspace varieties.** Let $Sub_{b_1,\dots,b_n} \subset \mathbb{P}(A_1^* \otimes \cdots \otimes A_n^*)$ denote the set of tensors T such that there exists subspaces $B_j \subseteq A_j^*$ with $\dim B_j = b_j$ and $T \in B_1 \otimes \cdots \otimes B_n$. $Sub_{b_1,\dots,b_n}$ is Zariski closed and its ideal is easy to describe. $I_d(Sub_{b_1,\dots,b_n})$ is the direct sum of the modules $S_{\mu_1}A_1 \otimes \cdots \otimes S_{\mu_n}A_n$ such that $S_{\mu_1}A_1 \otimes \cdots \otimes S_{\mu_n}A_n \subset S^d(A_1 \otimes \cdots \otimes A_n)$ and the length of some μ_j is greater than b_j. (In [4] we prove the generators of the ideal are indeed the expected ones.)

Assuming all the b_j are equal to say b_0, then $Sub_{b_0,\dots,b_0}$ is defined by equations of degree b_0+1, namely all the modules in $S^{b_0+1}(A_1 \otimes \cdots \otimes A_n)$ containing an exterior power of some A_j. In other words, as a set, $Sub_{r,\dots,r}$ is the intersection of all the r-th secant varieties of flattenings of the form $A_i \otimes (A_1 \otimes \cdots \otimes \hat{A}_i \otimes \cdots \otimes A_n)$.

In particular, $\sigma_r(\mathbb{P}A_1^* \times \cdots \times \mathbb{P}A_n^*) \subset Sub_{b_1,\dots,b_n}$ for all $b_1, \dots, b_n$ with $b_i \geq r$. We summarize the above discussion:

Proposition 2.2. *Defining equations for $\sigma_r(\mathbb{P}A_1^* \times \cdots \times \mathbb{P}A_n^*)$, when $\dim A_j^* \geq r$ may be obtained from the union of the the modules inherited from defining equations for $\sigma_r(\mathbb{P}^{r-1} \times \cdots \times \mathbb{P}^{r-1})$ and defining equations for $Sub_{r,\dots,r}$.*

Remark 2.3. For ordinary matrices, i.e., points in the tensor product of two vector spaces, there is just one notion of rank, but it has several generalizations to tensor products of several vector spaces. The first is the minimum number of monomials required to express a given tensor as a sum of monomials, which is now commonly called the *rank* of the tensor. The second is the smallest secant variety of the Segre variety in which the tensor lies, which is called the *border rank* of the tensor. A third notion comes from *Cayley's hyperdeterminant*, a higher dimensional generalization of the determinant. Already for $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ this notion diverges from the previous two, in the sense that every tensor in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ has border rank at most two, but the zero set of the hyperdeterminant is a quartic hypersurface. For $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ the hyperdeterminant describes an irreducible hypersurface of degree 36 whereas $\sigma_4(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2)$ is a hypersurface of degee 9. The hyperdeterminants induce "hyper-minors" by inheriting the corresponding modules, but the zero sets of these appear to have little relation with secant varieties. A fourth notion generalizes to the *subspace varieties*, because $T \in A \otimes B$ has rank r iff there exist $A' \subset A$, $B' \subset B$, both of dimension r, with $T \in A' \otimes B'$. Tensors of border rank r are in general only contained in $Sub_{r,\dots,r}$.

3. Strassen's equations

3.1. **Strassen's theorem.** For a tensor $T \in A \otimes B \otimes C$ and $\alpha \in A^*$, let $T_\alpha \in B \otimes C$ denote the contraction of T with α.

Theorem 3.1 (Strassen)**.** [7] *Let $3 \leq a \leq b = c \leq r$. Let $T \in A\otimes B\otimes C$ and $\alpha \in A^*$ be such that rank$T_\alpha = b$. For all $\alpha^1, \alpha^2 \in A^*$, consider the linear maps $T_{\alpha,\alpha^j} : B \to B$ by considering $T_\alpha : C^* \to B$ and $T_{\alpha,\alpha^j} = T_{\alpha^j}{T_\alpha}^{-1}$. If $[T] \in \sigma_r(\mathbb{P}A\times\mathbb{P}B\times\mathbb{P}C)$, then*

$$\text{rank}\,[T_{\alpha,\alpha^1}, T_{\alpha,\alpha^2}] \leq 2(r-b).$$

Moreover for a generic tensor $T \in A\otimes B\otimes C$, $[T_{\alpha,\alpha^1}, T_{\alpha,\alpha^2}]$ is of maximal rank.

This theorem (together with an easy application of Terracini's lemma) implies $\sigma_4(\mathbb{P}^2\times\mathbb{P}^2\times\mathbb{P}^2)$ is a hypersurface. It also implies that the border rank of the multiplication of $m\times m$ matrices is at least $\frac{3m^2}{2}$. Here is a proof that is essentially Strassen's, rephrased more invariantly to enable generalizations.

Proof. First note that it is sufficient to prove the result for T of the form $T = a_1b_1c_1+\cdots+a_rb_rc_r$ as these form a Zariski open subset of the irreducible variety σ_r. Here $a_j \in A$ etc... and $a_jb_jc_j = a_j\otimes b_j\otimes c_j$. Fix an auxiliary vector space $D \simeq \mathbb{C}^r$ and write $T_\alpha : C^* \to B$ as a composition of maps

$$C^* \xrightarrow{\ i\ } D \xrightarrow{\ \delta_\alpha\ } D \xrightarrow{\ p\ } B.$$

To see this explicitly, if $T = a_1b_1c_1+\cdots+a_rb_rc_r$ and we assume $b_1,...,b_b$, $c_1,...,c_b$ are bases of B, C, then letting $d_1,...,d_r$ be a basis of D, we have $i(\eta) = \sum_{j=1}^r \eta(c_j)d_j$, $\delta_\alpha(d_j) = \alpha(a_j)d_j$, for $1\leq s\leq b$ we have $p(d_s) = b_s$, and for $b+1\leq x\leq r$, writing $b_x = \xi_x^s b_s$, then we have $p(d_x) = \xi_x^s b_s$.

Let $D' = i(C^*)$, write $i' : C^* \to D'$ and set $p_\alpha := p\mid_{\delta_\alpha(D')}$, so $p_\alpha : \delta_\alpha(D') \to B$ is a linear isomorphism. Then we may write ${T_\alpha}^{-1} = (i')^{-1}{\delta_\alpha}^{-1}{p_\alpha}^{-1}$.

Note that rank$\,[T_{\alpha,\alpha^1}, T_{\alpha,\alpha^2}] = $ rank$\,(T_{\alpha^1}{T_\alpha}^{-1}T_{\alpha^2} - T_{\alpha^2}{T_\alpha}^{-1}T_{\alpha^1})$ because T_α is invertible. We have

$$\begin{aligned}
&T_{\alpha^1}{T_\alpha}^{-1}T_{\alpha^2} - T_{\alpha^2}{T_\alpha}^{-1}T_{\alpha^1}\\
&\quad= (p\delta_{\alpha^1}i')((i')^{-1}{\delta_\alpha}^{-1}{p_\alpha}^{-1})(p\delta_{\alpha^2}i') - (p\delta_{\alpha^2}i')((i')^{-1}{\delta_\alpha}^{-1}{p_\alpha}^{-1})(p\delta_{\alpha^1}i')\\
&\quad= p[\delta_{\alpha^1}{\delta_\alpha}^{-1}{p_\alpha}^{-1}p\delta_{\alpha^2} - \delta_{\alpha^2}{\delta_\alpha}^{-1}{p_\alpha}^{-1}p\delta_{\alpha^1}]i'\\
&\quad= p{\delta_\alpha}^{-1}[\delta_{\alpha^1}{p_\alpha}^{-1}p\delta_{\alpha^2} - \delta_{\alpha^2}{p_\alpha}^{-1}p\delta_{\alpha^1}]i'
\end{aligned}$$

where the last equality holds because the δ_α's commute.

Now ${p_\alpha}^{-1}p\mid_{\delta_\alpha(D')} = Id$, so write $D = \delta_\alpha(D')\oplus D''$, where we choose any complement to $\delta_\alpha(D')$ in D. We have $\dim D'' = r-b$ and we may write ${p_\alpha}^{-1}p = Id_{\delta_\alpha(D')} + f$ for some map $f : D'' \to D$. Thus

$$T_{\alpha^1}{T_\alpha}^{-1}T_{\alpha^2} - T_{\alpha^2}{T_\alpha}^{-1}T_{\alpha^1} = p{\delta_\alpha}^{-1}[\delta_{\alpha^1}f\delta_{\alpha^2} - \delta_{\alpha^2}f\delta_{\alpha^1}]i'$$

and is therefore of rank at most $2(r-b)$. □

3.2. **Towards a more invariant formulation of Strassen's theorem.** As stated, there are several undesirable aspects to Strassen's equations: the choices of $\alpha, \alpha^1, \alpha^2$, the requirement that α is such that T_α invertible, and the way the equations are written makes it difficult to see what equations will be inherited from them when we increase the dimensions of the spaces. Moreover, say $a = b = c$, then we can clearly change the roles of the spaces - are the new equations so obtained redundant or not?

A first step towards resolving these issues is to reconsider matrix multiplication and inverses more invariantly.

For a linear map $f : V \to W$, let $f^{\wedge k} : \Lambda^k V \to \Lambda^k W$ denote the induced linear map. If $\dim V = \dim W = n$ and f is invertible, then as a tensor $f^{\wedge n-1} = (f^{-1})^t\otimes\det(f)$. Recall that for a vector space of dimension n, that $\Lambda^{n-1}V = V^*\otimes\Lambda^n V$, so if V, W have dimension n, then

$\Lambda^{n-1}V^* \otimes \Lambda^{n-1}W = \text{Hom}\,(W,V)\otimes \Lambda^b V \otimes \Lambda^b W$. $f^{\wedge n-1}$ has the advantage over f^{-1} of being defined even if f is not invertible.

For $T \in A\otimes B\otimes C$, let $T^\alpha := T_\alpha^{\wedge b-1} \in \Lambda^{b-1}B\otimes\Lambda^{b-1}C = \Lambda^{b-1}B\otimes C^*\otimes\Lambda^b C$. We may contract $T^\alpha\otimes T_{\alpha^j}\in \Lambda^{b-1}B\otimes C^*\otimes\Lambda^b C\otimes B\otimes C$ to an element

$$T^\alpha_{\alpha^j}\in \Lambda^b B\otimes C^*\otimes \Lambda^b C\otimes C = C^*\otimes C\otimes\Lambda^b B\otimes\Lambda^b C.$$

Now consider

$$T^\alpha_{\alpha^1}\otimes T^\alpha_{\alpha^2}\in C^*\otimes C\otimes C^*\otimes C\otimes(\Lambda^b B)^{\otimes 2}\otimes(\Lambda^b C)^{\otimes 2}$$

and contract on the second and third factors to obtain an element of $C^*\otimes C\otimes(\Lambda^b B)^{\otimes 2}\otimes(\Lambda^b C)^{\otimes 2}$. This contraction of course corresponds to matrix multiplication, as does contraction in the first and fourth factor, which corresponds to multiplying the matrices in the opposite order. We do both contractions and take their difference and call the result

$$[T^\alpha_{\alpha^1},T^\alpha_{\alpha^2}]\in C^*\otimes C\otimes(\Lambda^b B)^{\otimes 2}\otimes(\Lambda^b C)^{\otimes 2}.$$

Strassen's theorem states that the rank of $[T^\alpha_{\alpha^1},T^\alpha_{\alpha^2}]$ is at most $2(r-b)$.

Equivalent to Strassen's observation that $\text{rank}\,[T_{\alpha,\alpha^1},T_{\alpha,\alpha^2}] = \text{rank}\,(T_{\alpha^1}{T_\alpha}^{-1}T_{\alpha^2} - T_{\alpha^2}{T_\alpha}^{-1}T_{\alpha^1})$, we can get away with a lower degree tensor by just contracting once with T^α to get elements of $B\otimes C\otimes\Lambda^b B\otimes\Lambda^b C$.

To eliminate the choices of α,α_1,α_2, we may consider the tensor T^α without having chosen α as $T^{(\cdot)}\in S^{b-1}A\otimes\Lambda^{b-1}B\otimes\Lambda^{b-1}C$, which is obtained as the projection of $(A\otimes B\otimes C)^{\otimes b-1}$ to the subspace $S^{b-1}A\otimes\Lambda^{b-1}B\otimes\Lambda^{b-1}C$. Similarly $T_{\alpha^j}\in B\otimes C$, may be thought of as $T_{(\cdot)}\in A\otimes B\otimes C$. We then contract

$$T^{(\cdot)}\otimes T_{(\cdot)}\otimes T_{(\cdot)}\in\Lambda^{b-1}B\otimes\Lambda^{b-1}C\otimes B\otimes C\otimes B\otimes C\otimes(S^{b-1}A\otimes A\otimes A)$$

in two different ways, first contracting the first factor with the third and the second with the sixth to obtain an element of $B\otimes C\otimes(S^{b-1}A\otimes A\otimes A)\otimes\Lambda^b B\otimes\Lambda^b C$, then contracting the first with the fifth and the second with the fourth. We then take the difference of the two to obtain an element of $B\otimes C\otimes\Lambda^b B\otimes\Lambda^b C\otimes(S^{b-1}A\otimes A\otimes A)$. Call the resulting tensor $\phi(T)$, i.e.,

$$\phi\in(A^*\otimes B^*\otimes C^*)^{\otimes b+1}\otimes(S^{b-1}A\otimes\Lambda^2 A)\otimes\Lambda^b B\otimes B\otimes\Lambda^b C\otimes C$$

and in fact descends to be an element of $S^{b+1}(A^*\otimes B^*\otimes C^*)\otimes(S^{b-1}A\otimes\Lambda^2 A)\otimes\Lambda^b B\otimes B\otimes\Lambda^b C\otimes C$.

The proof of Strassen's theorem may be rephrased in this language. We leave this as an entertaining exercise for the reader. (Hint: the Plücker relations for the Grassmannian $G(2,r)$ furnish the key to showing the bound on the rank of the commutator.)

3.3. **Strassen's equations as modules.** We first determine which modules in

$$\Lambda^2 A\otimes S^{b-1}A\otimes\Lambda^b B\otimes B\otimes C\otimes\Lambda^b C$$

map nontrivially into $S^{b+1}(A\otimes B\otimes C)$, when we use ϕ to compose the inclusion

$$\Lambda^2 A\otimes S^{b-1}A\otimes\Lambda^b B\otimes B\otimes C\otimes\Lambda^b C\subset(A\otimes B\otimes C)^{\otimes b+1}$$

with the projection $(A\otimes B\otimes C)^{\otimes b+1}\to S^{b+1}(A\otimes B\otimes C)$.

Since here $b=\dim B=\dim C$, we have

$$\Lambda^2 A\otimes S^{b-1}A\otimes\Lambda^b B\otimes B\otimes C\otimes\Lambda^b C = (S_{b,1}A\oplus S_{b-1,1,1}A)\otimes\Lambda_{b,1}B\otimes\Lambda_{b,1}C$$

so there are two possible modules. By [4], $S_{b,1}A\otimes\Lambda_{b,1}B\otimes\Lambda_{b,1}C$ does not occur in the ideal of $\sigma_r(\mathbb{P}^1\times\mathbb{P}^{b-1}\times\mathbb{P}^{c-1})$ so by inheritance it cannot occur when $\dim A>2$ either, so we are reduced to a unique module.

Taking minors corresponds to taking exterior powers in the B,C factors and we conclude:

Proposition 3.2. *As modules, the equations that imply*

$$\mathit{rank}[T^{\alpha}_{\alpha^1}, T^{\alpha}_{\alpha^2}] \leq 2(r-b).$$

for all choices of $\alpha, \alpha_1, \alpha_2 \in A^*$ correspond to the image of the inclusion via ϕ of

$$S^{2(r-b)+1}(S_{b-1,1,1}A) \otimes \Lambda^{2(r-b)+1}(\Lambda_{b,1}B) \otimes \Lambda^{2(r-b)+1}(\Lambda_{b,1}C)$$

into $S^{(2(r-b)+1)(b+1)}(A\otimes B\otimes C)$. When $r=b$ we obtain the single module

$$S_{b-1,1,1}A \otimes \Lambda_{b,1}B \otimes \Lambda_{b,1}C.$$

For $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ we obtain the same modules regardless of which factor we use to make the projections - all are the same copy of $S_{211}A \otimes S_{211}B \otimes S_{211}C$ for the case of σ_3 and of $S_{333}A \otimes S_{333}B \otimes S_{333}C$ for the case of σ_4. This redundancy fails for larger dimensional projective spaces.

3.4. **Example of Strassen's equations.** We write down a basis of the modules of polynomials corresponding to $S_{b-1,1,1}A \otimes \Lambda_{b,1}B \otimes \Lambda_{b,1}C$. Let $\alpha_i, \alpha_j, \alpha_k \in A^*$, let $\beta_1, ..., \beta_b$, $\xi_1, ..., \xi_b$ be bases of B^*, C^*. Consider the tensor

$$P^{i,j|k}_{s,t} = \alpha_i \wedge \alpha_j \otimes (\alpha_k)^{b-1} \otimes \beta_1 \wedge \cdots \wedge \beta_b \otimes \beta_s \otimes \xi_1 \wedge \cdots \wedge \xi_b \otimes \xi_t$$

Applying ϕ we obtain (ignoring scalars)

$$\begin{aligned}&(\alpha_2\otimes\alpha_3 - \alpha_3\otimes\alpha_2)\otimes(\alpha_1)^{b-1}\otimes(\sum_j (-1)^{j+1}\beta_{\hat{\jmath}}\otimes\beta_j\otimes\beta_s)\otimes(\sum_k(-1)^{k+1}\xi_{\hat{k}}\otimes\xi_k\otimes\beta_t)\\ &= (-1)^{j+k}[((\alpha_1)^{b-1}\otimes\beta_{\hat{\jmath}}\otimes\xi_{\hat{k}})\otimes(\alpha_2\otimes\beta_j\otimes\xi_t)\otimes(\alpha_3\otimes\beta_s\otimes\xi_k)\\ &\quad - ((\alpha_1)^{b-1}\otimes\beta_{\hat{\jmath}}\otimes\xi_{\hat{k}})\otimes(\alpha_3\otimes\beta_j\otimes\xi_t)\otimes(\alpha_2\otimes\beta_s\otimes\xi_k)].\end{aligned}$$

If we choose dual bases for A, B, C and write

$$T = \sum_l a_l \otimes X_l$$

where the a_l are dual to the α_l and X_l are represented as $b\times b$ matrices with respect to the dual bases of B, C, then

$$P^{i,j|k}_{s,t}(T) = \sum_{u,v}(-1)^{u+v}(\det X^{\hat{u}}_{k,\hat{v}})(X^u_{i,t}X^s_{j,v} - X^s_{i,v}X^u_{j,t})$$

where $X^{\hat{u}}_{j,\hat{v}}$ is X_j with its u-th row and v-th column removed.

4. Generalizations of Strassen's conditions

We now generalize Strassen's equations using our new perspective. Recall that the key point for Strassen's equations was that contracting a tensor $T \in A\otimes B\otimes C$ in two different ways yielded tensors that almost commute when $T \in \sigma_r$.

Consider, for s, t such that $s+t \leq b$ and $\alpha, \alpha_j \in A^*$, the tensors

$$T^{\wedge s}_{\alpha_j} \in \Lambda^s B \otimes \Lambda^s C,\ T^{\wedge t}_{\alpha} \in \Lambda^t B \otimes \Lambda^t C$$

(our old case was $s=1, t=b-1$). We may contract $T^{\wedge t}_{\alpha} \otimes T^{\wedge s}_{\alpha_1} \otimes T^{\wedge s}_{\alpha_2}$ to obtain elements of $\Lambda^{s+t}B \otimes \Lambda^{s+t}C \otimes \Lambda^s B \otimes \Lambda^s C$ in two different ways, call these contractions $\psi^{s,t}_{\alpha,\alpha_1,\alpha_2}(T)$ and $\psi^{s,t}_{\alpha,\alpha_2,\alpha_1}(T)$.

Now say we may write $T = a_1 \otimes b_1 \otimes c_1 + \cdots + a_r \otimes b_r \otimes c_r$ for elements $a_i \in A$, $b_i \in B$, $c_i \in C$. We have

$$\psi^{s,t}_{\alpha,\alpha_1,\alpha_2}(T) = \sum_{|I|=s,|J|=t,|K|=s} \langle a_I, \alpha_1\rangle\langle a_J, \alpha\rangle\langle a_K, \alpha_2\rangle(b_{I+J} \otimes b_K) \otimes (c_I \otimes c_{J+K}),$$

where we used the notation $a_{I+J} = a_I \wedge a_J$ etc.. For this to be nonzero, we need I and J to be disjoint subsets of $\{1, \ldots, r\}$. Similarly, J and K must be disjoint. If $s + t = r$ this implies $I = K$. We conclude:

Proposition 4.1. *For $T \in \sigma_{s+t}(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$, for all $\alpha, \alpha_1, \alpha_2 \in A^*$*

$$\psi^{s,t}_{\alpha,\alpha^1,\alpha^2}(T) - \psi^{s,t}_{\alpha,\alpha^2,\alpha^1}(T) = 0.$$

We have the bilinear map

$$(\Lambda^2(S^sA) \otimes S^tA)^* \times (A \otimes B \otimes C)^{\otimes 2s+t} \to \Lambda^{s+t}B \otimes \Lambda^{s+t}C \otimes \Lambda^sB \otimes \Lambda^sC.$$

whose image is $\psi^{s,t}_{\alpha,\alpha^1,\alpha^2}(T) - \psi^{s,t}_{\alpha,\alpha^2,\alpha^1}(T)$. We rewrite it as a polynomial map

$$\Psi^{s,t} : A \otimes B \otimes C \to (\Lambda^2(S^sA) \otimes S^tA) \otimes \Lambda^{s+t}B \otimes \Lambda^{s+t}C \otimes \Lambda^sB \otimes \Lambda^sC.$$

If we want to consider polynomial equations on $A \otimes B \otimes C$, they are the image of the transpose of $\Psi^{s,t}$. So just as with Strassen's equations, we no longer need to make choices of elements of A^*. The only catch is we don't yet know whether or not $\Psi^{s,t}(T)$ is identically zero for all tensors T. This is addressed in §7.

We write $r = s + t$ and call the image of $\Psi^{s,r-s}$ the *(r, s)-coercive equations.*

The modules for the (r, s)-coercive equations are the irreducible submodules of

$$\Lambda^2(S^sA) \otimes S^{r-s}A \otimes \Lambda^rB \otimes \Lambda^sB \otimes \Lambda^sC \otimes \Lambda^rC$$

that map isomorphically into $S^{r+s}(A \otimes B \otimes C)$ under the transpose of $\Psi^{s,t}$. There are many such submodules and we can describe them explicitly (see the formulas (4) below), but there is no easy to implement formula for the decomposition of $S^{r+s}(A \otimes B \otimes C)$. There are certain modules that are easily seen to occur in both $S^{r+s}(A \otimes B \otimes C)$ and $\Lambda^2(S^sA) \otimes S^{r-s}A \otimes \Lambda^rB \otimes \Lambda^sB \otimes \Lambda^sC \otimes \Lambda^rC$, and we will show that at least most of the time, these modules map isomorphically, so that most of the (r, s)-coercive conditions lead to nontrivial equations.

Theorem 4.2. *For s odd, r even, and $2s \leq r$, or r, s odd and $3s \leq r$, the multiplicity one component of $S_{r+s}(A \otimes B \otimes C)$ of type $S_{r-s,s,s}A \otimes \Lambda_{r,s}B \otimes \Lambda_{r,s}C$ induced from the (r, s)-coercive equations is a nontrivial set of equations of $\sigma_r(\mathbb{P}A^* \times \mathbb{P}B^* \times \mathbb{P}C^*)$. All these modules of equations for σ_r are independent elements of the ideal of σ_r as s varies.*

The proof of nontriviality is given in §7. To see the independence, consider equations of degrees $r+s_1$ and $r+s_2$ with $s_1 < s_2$. Were the second set induced by the first, the corresponding tableau for $S_{r-s_1,s_1,s_1}A$ would have to fit inside the tableau for $S_{r-s_2,s_2,s_2}A$. But since $r - s_1 > r - s_2$ the first tableau has a longer first row than the second.

5. $Comm^r_A$

We study the special case $s = 1$. Then we have the decompositions

$$\Lambda^2A \otimes S^{r-1}A = S_{r,1}A \oplus S_{r-1,1,1}A,$$
$$B \otimes \Lambda^rB = \Lambda^{r+1}B \oplus \Lambda_{r,1}B,$$
$$C \otimes \Lambda^rC = \Lambda^{r+1}C \oplus \Lambda_{r,1}C.$$

We may ignore the modules containing an $(r+1)$-st exterior power as we already know all those are contained in the ideal, and we may eliminate $S_{r,1}A\otimes\Lambda_{r,1}B\otimes\Lambda_{r,1}C$ as above. Thus we are reduced to studying the modules inherited from Strassen's equations.

Definition 1. We let $Comm^r_A\subset\mathbb{P}(A\otimes B\otimes C)$ be the set of tensors T such that $\Psi^{1,r-1}(T)=0$. This set is Zariski closed whose ideal is generated by the image of the transpose of $\Psi^{1,r-1}$. The case $a=3$, $r=b=c$ corresponds to the tensors obeying Strassen's commutation condition $[T_{\alpha,\alpha_1},T_{\alpha,\alpha_2}]=0$ for all $\alpha,\alpha_1,\alpha_2\in A^*$ such that T_α is invertible.

Note that these equations are of the minimal degree $r+1$ (see [3]).

Proposition 5.1. *For $a=3\le r\le b,c$,*

$$Comm^r_A=\sigma_r(\mathbb{P}A^*\times\mathbb{P}B^*\times\mathbb{P}C^*)\cup Sub_{3,r-1,r}\cup Sub_{3,r,r-1}.$$

Proof. The set of defining equations for $Comm^r_A$ is $S_{211}A\otimes\Lambda_{r,1}B\otimes\Lambda_{r,1}C$. In particular they all involve terms containing partitions of length r in B and C, thus they vanish on $Sub_{3,r,r-1}\cup Sub_{3,r-1,r}$. We also already saw that $Comm^r_A\supseteq\sigma_r$, so we have

$$Comm^r_A\supseteq\sigma_r(\mathbb{P}A^*\times\mathbb{P}B^*\times\mathbb{P}C^*)\cup Sub_{3,r,r-1}\cup Sub_{3,r-1,r}.$$

Let $T\in Comm^r_A$ be such that $T\notin Sub_{3,r,r-1}\cup Sub_{3,r-1,r}$. Let $B'\subset B$, $C'\subset C$ be the smallest subspaces such that $T\in A\otimes B'\otimes C'$. $T\in Comm^r_A$ implies that B',C' both have dimension at most r and $T\notin Sub_{3,r,r-1}\cup Sub_{3,r-1,r}$, implies further that both have dimension exactly r.

Fix $\alpha_0\in A$ and consider the abelian subalgebra $\{T_{\alpha_0,\alpha_1},T_{\alpha_0,\alpha_2}\}\subset End(C')$. Now the crucial point is that any pair of commuting matrices can be approximated by simultaneously diagonalizable matrices. (This statement is the only place where we use the hypothesis that $a=3$. It is a slightly more precise statement than the well-known irreducibility of the commuting variety [6]. Note that the corresponding statement is not true for three or more commuting matrices.) That is, our tensor T is in the closure of the set of those T''s for which we can find a basis $b_1,\dots,b_r$ of B', and a basis $c_1,\dots,c_r$ of C', such that any $T'(\alpha)$ is a linear combination of $b_1\otimes c_1,\dots,b_r\otimes c_r$. But then we can find $a_1,\dots,a_r$ in A, such that $T'=a_1\otimes b_1\otimes c_1+\cdots+a_r\otimes b_r\otimes c_r$. In particular such a T' belongs to σ_r, hence so does T. $\square$

Now, since $\sigma_3(\mathbb{P}^2\times\mathbb{P}^2\times\mathbb{P}^1)$ is the entire ambient space, $Sub_{3,3,2}\cup Sub_{3,2,3}\subset\sigma_3(\mathbb{P}^{a-1}\times\mathbb{P}^{b-1}\times\mathbb{P}^{c-1})$ and we conclude:

Corollary 5.2. *As sets, for $a,b,c\ge 3$,*

$$\sigma_3(\mathbb{P}^{a-1}\times\mathbb{P}^{b-1}\times\mathbb{P}^{c-1})=Comm^3_A\cap Sub_{333}=Comm^3_B\cap Sub_{333}=Comm^3_C\cap Sub_{333}.$$

That is $\sigma_3(\mathbb{P}^{a-1}\times\mathbb{P}^{b-1}\times\mathbb{P}^{c-1})$ is the zero set of $S_{211}A\otimes S_{211}B\otimes S_{211}C\subset S^4(A\otimes B\otimes C)$ and modules in degree four containing a fourth exterior power (i.e., $\Lambda^4A\otimes\Lambda^4(B\otimes C)$ plus permuatations). In particular, σ_3 is cut out set-theoretically by equations of degree four.

Remark 5.3. In fact, the stronger statement that the ideal of σ_3 is generated by the above modules holds, see [4], but the proof relies on a computer calculation.

Proposition 5.4. *For $a\le b,c$ and $r\le 4$,*

$$Comm^r_A=\sigma_r(\mathbb{P}A^*\times\mathbb{P}B^*\times\mathbb{P}C^*)\cup Sub_{a,r-1,r}\cup Sub_{a,r,r-1}.$$

Proof. The proof is the same as above except that at the point where we used $a=3$ we use instead that for $r\le 4$, an r-dimensional abelian subalgebra of $\mathfrak{gl}_r$ can be approximated by Cartan subalgebras (subalgebras of matrices that are diagonal in some fixed basis) [2] and we conclude as above. $\square$

Remark 5.5. It is likely that 5-dimensional abelian subalgebra of $\mathfrak{gl}_5$ can be approximated by Cartan subalgebras so proposition 5.4 should still hold for $r = 5$, [2]. On the other hand, it is not possible to approximate r-dimensional abelian subalgebras of $\mathfrak{gl}_r$ by Cartan algebras for $r > 5$.

Corollary 5.6. *As sets, for $a, b, c \geq 3$, $\sigma_4(\mathbb{P}^{a-1} \times \mathbb{P}^{b-1} \times \mathbb{P}^{c-1})$ is the zero set of*

(1) $(S_{311}A \otimes S_{2111}B \otimes S_{2111}C) \oplus (S_{2111}A \otimes S_{311}B \otimes S_{2111}C) \oplus (S_{2111}A \otimes S_{2111}B \otimes S_{311}C) \subset S^5(A \otimes B \otimes C)$, *i.e., the equations of $Comm^4$.*
(2) *equations inherited from $\sigma_4(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3)$*
(3) *modules in $S^5(A \otimes B \otimes C)$ containing a fifth exterior power, i.e., the equations for $Sub_{4,4,4}$.*

Remark 5.7. The known defining modules for $\sigma_4(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3)$ are $S_{321}A \otimes S_{321}B \otimes S_{3111}C$ in degree 6 and $S_{333}A \otimes S_{333}B \otimes S_{333}C$ in degree 9, [3]. We do not have an interpretation for $S_{321}A \otimes S_{321}B \otimes S_{3111}C$, and it would be useful to have one in order to determine if the known modules for $\sigma_4(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3)$ are sufficient to define it. In [3] there is a typographical error in the statement of proposition 6.3, incorrectly giving the modules in degree six, although they are written correctly in the proof.

6. Coercive contractions

We now place the discussion of §4 in a more general context. Let m and k be integers, with m even. Consider the projection

$$S^k(A_1 \otimes \cdots \otimes A_m) \quad \longrightarrow \quad \Lambda^k A_1 \otimes \cdots \otimes \Lambda^k A_m,$$

sending $T = \sum_i a_1^i \otimes \cdots \otimes a_m^i$ to

$$\wedge^k T = \sum_{|I|=k} a_1^I \otimes \cdots \otimes a_m^I,$$

where if $I = (i_1 < \cdots < i_k)$, then $a^I = a^{i_1} \wedge \cdots \wedge a^{i_k}$.

Let

$$T = \sum_{1 \leq i \leq r} a_0^i \otimes \cdots \otimes a_m^i \in A_0^* \otimes \cdots \otimes A_m^*$$

for some vectors $a_j^i \in A_j^*$. For any $\alpha \in A_0$, let $T(\alpha) \in A_1^* \otimes \cdots \otimes A_m^*$ denote the contraction of T by α. Then

$$\wedge^k T(\alpha) = \sum_{|I|=k} \langle a_0^I, \alpha \rangle a_1^I \otimes \cdots \otimes a_m^I.$$

Now consider the product of p such tensors,

$$\begin{aligned} &\wedge^{k_1} T(\alpha_1) \otimes \cdots \otimes \wedge^{k_p} T(\alpha_p) \\ &= \sum_{|I_1|=k_1, \ldots, |I_p|=k_p} \langle a_0^{I_1}, \alpha_1 \rangle \cdots \langle a_0^{I_p}, \alpha_p \rangle (a_1^{I_1} \otimes \cdots \otimes a_1^{I_p}) \otimes \cdots \otimes (a_m^{I_1} \otimes \cdots \otimes a_m^{I_p}). \end{aligned} \tag{1}$$

Note that we put together the different terms involving wedge powers of each A_j^*. This is because we want to take more skew-symmetrizations, that is, we want to apply natural maps of type

$$\Lambda^{m_1} A_1^* \otimes \cdots \otimes \Lambda^{m_t} A_1^* \to \Lambda^{m_1 + \cdots + m_t} A_1^* \tag{2}$$

to our tensor.

Definition 2. A contraction

$$\Gamma : A_0^{\times p} \times (A_0^* \otimes \cdots \otimes A_m^*) \to (\Lambda^{k_1} A_1^* \otimes \cdots \otimes \Lambda^{k_1} A_m^*) \otimes \cdots \otimes (\Lambda^{k_p} A_1^* \otimes \cdots \otimes \Lambda^{k_p} A_m^*) \tag{3}$$

given by (1) followed by maps of the form (2) is *r-coercive* if when restricted to tensors of the form $T = a_0^1\otimes\cdots\otimes a_m^1 + \cdots + a_0^r\otimes\cdots\otimes a_m^r$ the only nonzero terms in the right hand side of (1) are terms with $I_1 = I_2$ (or more generally the only nonzero terms are those where two of the multi-indices I_j coincide). Generalizing our previous discussion, r-coercive contractions furnish equations for $\sigma_r(\mathbb{P}A_0^*\times\cdots\times\mathbb{P}A_m^*)$, by taking $(\Gamma-\Gamma')(T)$ where Γ' is the same as Γ only switching the roles of the coinciding multi-indices.

Such a contraction is called *partially r-coercive* if when restricted to tensors of the form $T = a_0^1\otimes\cdots\otimes a_m^1 + \cdots + a_0^r\otimes\cdots\otimes a_m^r$ the contracted tensor is nongeneric among tensors in the image of $\Gamma-\Gamma'$.

Strassen's tensors ϕ are partially r-coercive because the contracted tensor can have rank (as a matrix) at most $2(r-b)$ whereas a generic such matrix has rank b. The tensors $\Psi^{s,t}$ are $(s+t)$-coercive and partially $(s+t+x)$-coercive for small x.

Partially coercive tensors Γ applied to $T\in\sigma_r$ for sufficiently small r give rise to tensors $\Gamma(T)$ that belong to some type of secant variety. Since our understanding of higher secant varieties is quite limited in general, it is not always clear how to use them. However, there is one case we understand well, namely the secant varieties of two-factor Segre varieties, which is what is used for the Strassen equations.

Here is a more complicated example of a coercive contration:

Example 3. Let $m=6$ and $p=7$, $k_1=k_2=k_3=r-4s$ and $k_4=k_5=k_6=k_7=s$. Then the contraction $\psi_{145,167,246,257,347,356}$ is r-coercive. (Here the grouped indices indicate which are to be contracted together.) Indeed, the contraction 145 implies for the surviving terms that $I_1\cup I_4\cup I_5$ is a disjoint union, in other words I_4 and I_5 are disjoint and I_1 is contained in the complement of their union. Taking the other contractions into account, we see that I_4, I_5, I_6, I_7 are pairwise disjoint, and that I_1, I_2, I_3 are contained in, hence equal to because of the cardinalities, the complement of their union. In particular they must be equal.

Example 4. Here are some further examples of partially coercive equations, and further variants on these should be clear to the reader. In the propositions below we assume $b=c$. Of course the corresponding modules induce equations when this is not the case, but moreover when $b\le r\le c$ there are further modules of equations that are induced that are not inherited.

Proposition 6.1. *Let $T\in A^*\otimes B^*\otimes C^*$ and $\alpha_0,\alpha_1,\alpha,\alpha'\in A$. If $T\in\sigma_r(\mathbb{P}A^*\times\mathbb{P}B^*\times\mathbb{P}C^*)$, then*

$$rank\ [T_\alpha^{\alpha_0}, T_{\alpha'}^{\alpha_1}] \le 3(r-b).$$

The relevant modules are the corresponding image of $\Lambda^2(S^{b-1}A)\otimes\Lambda^2A\otimes\Lambda^bB\otimes B\otimes\Lambda^bC\otimes C$ in $S^{2b}(A\otimes B\otimes C)$.

Proposition 6.2. *Let $T\in A^*\otimes B^*\otimes C^*$ and $\alpha_0,\alpha_1,...,\alpha_k\in A$. If $T\in\sigma_r(\mathbb{P}A^*\times\mathbb{P}B^*\times\mathbb{P}C^*)$, then for any permutation $\sigma\in\mathfrak{S}_k$,*

$$rank\ \left(T_{\alpha_1}^{\alpha_0}\cdots T_{\alpha_k}^{\alpha_0} - T_{\alpha_{\sigma(1)}}^{\alpha_0}\cdots T_{\alpha_{\sigma(k)}}^{\alpha_0}\right) \le 2(k-1)(r-b).$$

The relevant modules are the corresponding image of

$$S^{k-1}(S^{b-1}A)\otimes\Lambda^kA\otimes(\Lambda^bB)^{\otimes k-1}\otimes B\otimes(\Lambda^bC)^{\otimes k-1}\otimes C$$

in $S^{(k-1)b+1}(A\otimes B\otimes C)$.

The proofs are similar to the proof of Strassen's theorem.

Another variant is obtained by using products of $T_{\alpha'}^{\alpha}$ with different α's and permuted α''s.

7. Nontriviality of the (r,s)-coersive equations

We study the image of

$$\Psi^{r,s}:\Lambda^2(S^sA)\otimes S^{r-s}A\otimes \wedge^r B\otimes \wedge^s B\otimes \wedge^s C\otimes \wedge^r C)\to S^{r+s}(A\otimes B\otimes C).$$

Recall that this may be thought of as first embeding $\Lambda^2(S^sA)\otimes S^{r-s}A\otimes\wedge^rB\otimes\wedge^sB\otimes\wedge^sC\otimes\wedge^r C$ in $(A\otimes B\otimes C)^{\otimes r+s}$ according the the recipe in §4 and then projecting to the symmetric algebra. We write the inclusion and projection as follows:

$$\begin{array}{c}
\Lambda^2(S^sA)\otimes S^{r-s}A\otimes \wedge^r B\otimes \wedge^s B\otimes \wedge^s C\otimes \wedge^r C\\
\downarrow\\
S^sA\otimes S^{r-s}A\otimes S^sA\otimes \wedge^s B\otimes \wedge^{r-s} B\otimes \wedge^s B\otimes \wedge^s C\otimes \wedge^{r-s} C\otimes \wedge^s C\\
||\\
S^sA\otimes \wedge^s B\otimes \wedge^s C\otimes S^{r-s}A\otimes \wedge^{r-s} B\otimes \wedge^{r-s} C\otimes S^sA\otimes \wedge^s B\otimes \wedge^s C\\
\downarrow\\
S^s(A\otimes B\otimes C)\otimes S^{r-s}(A\otimes B\otimes C)\otimes S^s(A\otimes B\otimes C)\\
\downarrow\\
S^{r+s}(A\otimes B\otimes C).
\end{array}$$

The first two maps are injective, the last one is surjective but not injective and the problem is to understand whether its kernel may contain the subspace we are interested in. For this we need to understand the above maps in detail, which are made of elementary maps that we write down explicitly.

First, we have the map

$$\begin{array}{ccc}
\wedge^r B & \hookrightarrow & \wedge^sB\otimes \wedge^{r-s} B\\
f_1\wedge\cdots\wedge f_r & \mapsto & \displaystyle\sum_{I=(i_1<\cdots<i_s)}\varepsilon(I,\hat I)f_{i_1}\wedge\cdots\wedge f_{i_s}\otimes f_{\hat\imath_1}\wedge\cdots\wedge f_{\hat\imath_{r-s}},
\end{array}$$

with the following notation: $\hat I=(\hat\imath_1<\cdots<\hat\imath_{r-s})$ is the complementary sequence to I in $(1,\dots,r)$, and $\varepsilon(I,\hat I)$ is the sign of the permutation $(1,\dots,r)\mapsto(I,\hat I)$ (a shuffle).

Second, we have the map

$$\begin{array}{ccc}
S^sA\otimes \wedge^s B\otimes \wedge^s C & \longrightarrow & S^s(A\otimes B\otimes C)\\
e^s\otimes f_1\wedge\cdots\wedge f_s\otimes g_1\wedge\cdots\wedge g_s & \mapsto & \displaystyle\sum_{\sigma\in\mathfrak{S}_s}\varepsilon(\sigma)(ef_1g_{\sigma(1)})\cdots(ef_sg_{\sigma(s)}).
\end{array}$$

In principle, this information is enough to check if a given irreducible component of

$$\Lambda^2(S^sA)\otimes S^{r-s}A\otimes \wedge^r B\otimes \wedge^s B\otimes \wedge^s C\otimes \wedge^r C$$

is mapped to zero, or to an isomorphic copy inside $S^{r+s}(A\otimes B\otimes C)$. And we just need to test this alternative on some highest weight vector.

Recall the decomposition formulas (e.g. [5]):

$$\begin{aligned}
\Lambda^2(S^sV)&=\bigoplus_{j:\mathrm{odd}}S_{2s-j,j}V\\
\Lambda^aV\otimes\Lambda^bV&=\bigoplus_{\substack{u+v=a+b\\ v\leq\min(a,b)}}\Lambda_{u,v}V\\
S_{a_1,a_2}V\otimes S_bV&=\bigoplus_{\substack{\rho+\sigma\leq b\\ a_2+\sigma\leq a_1}}S_{a_1+\rho,a_2+\sigma,b-\rho-\sigma}V
\end{aligned}\tag{4}$$

Since we don't have a closed form formula for $\Lambda_{a_1+\rho,a_2+\sigma,b-\rho-\sigma}(B\otimes C)$, or more precisely the factors in it of the form $\Lambda_{u,v}B\otimes\Lambda_{u',v'}C$ we cannot give a closed form formula for all the possible relevant factors appearing in $S^{r+s}(A\otimes B\otimes C)$. Even if we did have such a list, for any given module, we would still have to check that the resulting map was nonzero before concluding it was present.

We focus on cases that are of length three in A because those of length two are inherited from $\sigma_r(\mathbb{P}^1\times\mathbb{P}^{b-1}\times\mathbb{P}^{c-1})$ which is treated in [4].

For example, note that for s odd and $r\geq 2s$, $S_{r-s,s,s}A\subset S_{s,s}A\otimes S^{r-s}A$ with multiplicity one. Also $\wedge^r B\otimes\wedge^s B$ contains $\Lambda_{r,s}B$ with multiplicity one. We prove that the module $S_{r-s,s,s}A\otimes\Lambda_{r,s}B\otimes\Lambda_{r,s}C$ is not mapped to zero in $S^{r+s}(A\otimes B\otimes C)$ in many cases.

We write down a highest weight vector. The tensor product $f_1\wedge\cdots\wedge f_r\otimes f_1\wedge\cdots\wedge f_s$ gives a highest weight vector for $\Lambda_{r,s}B$ inside $\wedge^r B\otimes\wedge^s B$, where the f_i define a weight basis of B such that the ordering of the weights corresponds to the ordering of the indicies. Similarly $g_1\wedge\cdots\wedge g_r\otimes g_1\wedge\cdots\wedge g_s$ gives a highest weight vector for $\Lambda_{r,s}C$. To find a highest weight vector for $S_{r-s,s,s}A$ inside $S^{r-s}A\otimes S^sA\otimes S^sA$ we use Young symmetrizers [8]. The symmetrizer $c_{(r-s,s,s)}$ applied to each of the s factors of $A\otimes A\otimes A$ yields the highest weight vector

$$\Theta=\quad \textstyle\sum_{\sigma_1,\dots,\sigma_s\in\mathfrak{S}_3}\varepsilon(\sigma_1)\cdots\varepsilon(\sigma_s)e_1^{r-2s}e_{\sigma_1(1)}\cdots e_{\sigma_s(1)}\otimes e_{\sigma_1(2)}\cdots e_{\sigma_s(2)}\otimes e_{\sigma_1(3)}\cdots e_{\sigma_s(3)}.$$

where e_1,e_2,e_3 is an ordered weight basis for A and $\epsilon(\sigma)$ denotes the sign of the permutation σ. Considering the contributions of the six different permutations in $\mathfrak{S}_3$, we get

$$\Theta=\sum_{\alpha_1+\cdots+\alpha_6=s}(-1)^{\alpha_2+\alpha_4+\alpha_6}\binom{s}{\alpha}$$
$$e_1^{r-2s+\alpha_1+\alpha_2}e_2^{\alpha_3+\alpha_4}e_3^{\alpha_5+\alpha_6}\quad\otimes e_1^{\alpha_4+\alpha_5}e_2^{\alpha_1+\alpha_6}e_3^{\alpha_2+\alpha_3}\otimes e_1^{\alpha_3+\alpha_6}e_2^{\alpha_2+\alpha_5}e_3^{\alpha_1+\alpha_4}.$$

where $\binom{s}{\alpha}=\binom{s}{\alpha_1}\cdots\binom{s}{\alpha_6}$. Now we take the tensor product of our three highest weight vectors and examine the tensor Θ' that we get inside $S^{r+s}(A\otimes B\otimes C)$. To show this tensor is nonzero, we check that the coefficient of

$$(e_1f_1g_1)\cdots(e_1f_{r-s}g_{r-s})(e_2f_1g_r)\cdots(e_2f_sg_{r+1-s})(e_3f_rg_1)\cdots(e_3f_{r+1-s}g_s)$$

is nonzero. The contributions to this monomial in Θ' is the sum of the contributions from terms of the form

$$e_1^{r-2s+\alpha_1+\alpha_2}e_2^{\alpha_3+\alpha_4}e_3^{\alpha_5+\alpha_6}f_{\hat I}g_{\hat J}\otimes e_1^{\alpha_4+\alpha_5}e_2^{\alpha_1+\alpha_6}e_3^{\alpha_2+\alpha_3}f_{1\cdots s}g_J\otimes e_1^{\alpha_3+\alpha_6}e_2^{\alpha_2+\alpha_5}e_3^{\alpha_1+\alpha_4}f_Ig_{1\cdots s},$$

with some coefficient. The first (resp. second, third) of the three terms in this product will contribute to Θ' by a product of terms of the form $(e_if_jg_k)$, where for each given i, the index k describes a set A_i (resp. B_i, C_i), with

$$\begin{aligned}A_1\cup A_2\cup A_3&=\hat J\\ B_1\cup B_2\cup B_3&=J\\ C_1\cup C_2\cup C_3&=\{1,\dots,s\}.\end{aligned}$$

To contribute to our preferred monomial, we also need the conditions

$$\begin{aligned}A_1\cup B_1\cup C_1&=\{1,\dots,r-s\}\\ A_2\cup B_2\cup C_2&=\{1,\dots,s\}\\ A_3\cup B_3\cup C_3&=\{r-s+1,\dots,r\}.\end{aligned}$$

Now consider the index j in the different terms $(e_i f_j g_k)$. We need $j = k$ if $k \leq r - s$, and $j = \bar{k} := r + 1 - k$ otherwise. This leads to one more set of identities,

$$\begin{aligned} A_1 \cup \bar{A}_2 \cup \bar{A}_3 &= \hat{I} \\ B_1 \cup \bar{B}_2 \cup \bar{B}_3 &= \{1, \ldots, s\}, \\ C_1 \cup \bar{C}_2 \cup \bar{C}_3 &= I, \end{aligned}$$

where $\bar{A}$ denotes the image of A by the map $k \mapsto \bar{k}$. Note that all these unions are between pairwise disjoint sets.

The first two relations involving C_3 imply that $C_3 = \emptyset$. Since $\bar{B}_2 \subset \{1, \ldots, s\}$, we deduce that $B_2 = \emptyset$, hence $C_1 = A_2$. In particular, $\alpha_1 = \alpha_4 = \alpha_6 = 0$. Since also $\bar{B}_1 \subset \{1, \ldots, s\}$, we get that $A_1 = A_0 \cup \{s+1, \ldots, r-s\}$ where A_0 is the complement to $A_2 \cup B_1$ inside $\{1, \ldots, s\}$. Comparing I and $\hat{I}$ we deduce that $\bar{A}_3 = B_1$, hence $A_3 = \bar{B}_1$ and $B_3 = \bar{A}_0 \cup \bar{A}_2$. In particular, I and J are determined by A_0 and A_2. Note that once we have I and J, we can easily compute the signs $\varepsilon(I, \hat{I})$ and $\varepsilon(J, \hat{J})$. The result is that

$$\varepsilon(I, \hat{I})\varepsilon(J, \hat{J}) = (-1)^{(s+\alpha_2)(r-s+\alpha_2)}.$$

We deduce that the total contribution to our monomial is

$$T_{s,r-2s} := \sum_{\alpha_2+\alpha_3+\alpha_5=s} (-1)^{\alpha_2+(s+\alpha_2)(r-s+\alpha_2)} \binom{s}{\alpha}^2 (r-2s+\alpha_2)!\alpha_3!\alpha_5!\alpha_5!(\alpha_2+\alpha_3)!\alpha_3!(\alpha_2+\alpha_5)!.$$

Indeed, for a given α we have $\binom{s}{\alpha}$ choices for A_0, A_2, and once these are fixed, the number of permutations sending the e_i's to the g_k such that $k \in A_i$ is $\#A_1!\#A_2!\#A_3! = (r-2s+\alpha_2)!\alpha_3!\alpha_5!$, and so on.

So what remains to prove is that $T_{s,r-2s} \neq 0$. Observe that since s is odd, the product $(s+\alpha_2)\alpha_2$ is even. So

$$(-1)^{\alpha_2+(s+\alpha_2)(r-s+\alpha_2)} = \begin{cases} (-1)^{\alpha_2} & \text{for } r-s \text{ even} \\ -1 & \text{for } r-s \text{ odd.} \end{cases}$$

In particular, $T_{s,r-2s}$ is nonzero for r even. We are not able to prove all the remaining cases, but we are able to show:

Lemma 5. *The integer*

$$T_{s,t} = \frac{1}{(s!)^2} \sum_{\alpha+\beta+\gamma=s} (-1)^\alpha \frac{(\alpha+t)!(\alpha+\beta)!(\alpha+\gamma)!}{\alpha!\alpha!}$$

is nonzero for s, t odd and $t \geq s$.

Proof. Write $s = 2m+1$.

$$\begin{aligned} (s!)^2 T_{s,t} &= -\sum_{p=0}^{m} \sum_{\beta=0}^{2m+1-(2p+1)} \frac{(2p+1+t)!(2p+1+\beta)!(2m+1-\beta)!}{(2p+1)!(2p+1)!} + \sum_{p=0}^{m} \sum_{\beta=0}^{2m+1-2p} \frac{(2p+t)!(2p+\beta)!(2m+1-\beta)!}{(2p)!(2p)!} \\ &= -\sum_{p=0}^{m} \{ \sum_{\beta=0}^{2m+1-(2p+1)} \frac{(2p+1+t)(2p+t)!(2p+1+\beta)(2p+\beta)!(2m+1-\beta)!}{(2p+1)(2p)!(2p+1)(2p)!} \\ &\quad - \sum_{\beta=0}^{2m+1-(2p+1)} \frac{(2p+t)!(2p+\beta)!(2m+1-\beta)!}{(2p)!(2p)!} - \frac{(2p+t)!(2m+1)!(2p)!}{(2p)!(2p)!} \} \\ &= -\sum_{p=0}^{m} \{ \sum_{\beta=0}^{2m+1-(2p+1)} [\frac{(2p+t)!(2p+\beta)!(2m+1-\beta)!}{(2p)!(2p)!} (\frac{(2p+1+t)(2p+1+\beta)}{(2p+1)(2p+1)} - 1)] - \frac{(2p+t)!(2m+1)!(2p)!}{(2p)!(2p)!} \} \\ &= -\sum_{p=0}^{m} \frac{(2p+t)!}{(2p)!} \{ \sum_{\beta=0}^{2m+1-(2p+1)} [\frac{(2p+\beta)!(2m+1-\beta)!}{(2p+1)^2(2p)!} ((2p+1)(t+\beta)+t\beta)] - (2m+1)! \} \end{aligned}$$

In the case $t \geq 2m+1$ this gives the result immediately just by looking at the $\beta = 0$ term in the summation and noting it is not the only term. $\square$

We expect $T_{s,t}$ to be always nonzero when $s, t > 1$ and both are odd, but were unable to prove it. Note that it would be sufficient to prove the case $t = 1$ if we could show we always have the same sign.

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